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THE DIRICHLET PROBLEM FOR THE BRINKMAN SYSTEM IN SOBOLEV SPACES

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ABSTRACT. The Dirichlet problem for the Brinkman system and the Darcy-Forchheimer-Brinkman system are studied in $W^{s,q}(\Omega, \mathbb{R}^m) \times W^{s-1,q}(\Omega)$ for bounded domains $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary.

1. INTRODUCTION

The paper is devoted to the Dirichlet problem for the Brinkman system

$$-\Delta \mathbf{u} + \lambda \mathbf{u} + \nabla p = \mathbf{f}, \ \nabla \cdot \mathbf{u} = \chi \quad \text{in } \Omega \tag{1.1}$$

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega \tag{1.2}$$

and for the Darcy-Forchheimer-Brinkman system

$$-\Delta \mathbf{u} + \lambda \mathbf{u} + a|\mathbf{u}|\mathbf{u} + b(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \ \nabla \cdot \mathbf{u} = \chi \quad \text{in } \Omega. \tag{1.3}$$

Instead of the Dirichlet problem we shall study a bit more general nonlocal boundary condition

$$\mathbf{u} + \beta \int_\Omega \mathbf{u} \, \mathrm{d}x = \mathbf{g} \quad \text{on } \partial\Omega. \tag{1.4}$$

The problem is studied in Sobolev spaces $W^{s,q}(\Omega, \mathbb{R}^m) \times W^{s-1,q}(\Omega)$ in bounded domains with Lipschitz boundary. Here $1 \le s < \infty$ and $1 < q < \infty$. The boundary might be disconnected.

The Dirichlet problem for the Brinkman system in Sobolev spaces was studied in the following papers: [17] proves the existence of a solution in $H^{s+1/2}(\Omega; \mathbb{R}^m) \times H^{s-1/2}(\Omega)$ for $0 < s < 1$ and a bounded domain $\Omega \subset \mathrm{R}^m$ with connected Lipschitz boundary. The same result was proved in [29] for a bounded domain $\Omega \subset \mathrm{R}^m$ with Lipschitz boundary formed by two components. [12] is devoted to solutions in $W^{2,q}(\Omega; \mathbb{R}^m) \times W^{1,q}(\Omega)$ for a bounded domain with smooth boundary and $1 < q < \infty$. The same problem is studied in [9] and [10] for a bounded domain $\Omega \subset \mathbb{R}^m$ with boundary of class $\mathcal{C}^{1,1}$. Y. Shibata studies this problem in [31] for domains with boundary formed by two components.

The papers [14] and [15] studied the Dirichlet problem for the homogeneous Darcy-Forchheimer-Brinkman system in $W^{s,2}(\Omega, \mathbb{R}^m) \times W^{s-1,2}(\Omega)$, where $1 \le s < 3/2$, $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary and $m = 2$ or $m = 3$. The same problem was studied in [29] for domains which boundary is

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formed by two components. They supposed that a and b are positive constants, $\mathbf{f} \equiv 0$, $\chi \equiv 0$ and

$$\int_S \mathbf{g} \cdot \mathbf{n}^\Omega \, d\sigma = 0$$

for each component S of $\partial\Omega$.

In this paper we study the Brinkman system (1.1) in bounded domains $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary. Instead of the Dirichlet condition (1.2) we have a bit more general nonlocal boundary condition (1.4). We find a necessary and sufficient condition for the existence of a solution in $W^{s,q}(\Omega, \mathbb{R}^m) \times W^{s-1,q}(\Omega)$ with $1 \le s < \infty$, $1 < q < \infty$ in the following cases:

(1) $s = 1$ and $q = 2$.
(2) $\Omega \subset \mathbb{R}^2$, $s = 1$ and $4/3 < q < 4$.
(3) $\Omega \subset \mathbb{R}^3$, $s = 1$ and $3/2 < q < 3$.
(4) $\partial\Omega$ is of class $\mathcal{C}^1$ and $s = 1$.
(5) $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ with $k \in N$ and $s \le k + 1$.

We show that the velocity $\mathbf{u}$ is unique and the pressure p is unique up to an additive constant. Then we get results for the Darcy-Forchheimer-Brinkman system from the results for the Brinkman system using the fixed point theorem.

2. Function spaces

First we remember definitions of several function spaces.

Let $\Omega \subset \mathbb{R}^m$ be an open set. We denote by $\mathcal{C}_c^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. If $k \in \mathbb{N}_0$, $1 < q < \infty$ we define the Sobolev space $W^{k,q}(\Omega) := \{f \in L^q(\Omega); \partial^\alpha f \in L^q(\Omega) \text{ for } |\alpha| \le m\}$ endowed with the norm

$$\|u\|_{W^{k,q}(\Omega)} = \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^q(\Omega)}.$$

(Clearly $W^{0,q}(\Omega) = L^q(\Omega)$.) If $s = k + \lambda$, $0 < \lambda < 1$ and $1 < q < \infty$ denote $W^{s,q}(\Omega) := \{u \in W^{k,q}(\Omega); \|u\|_{W^{s,q}(\Omega)} < \infty\}$ where

$$\|u\|_{W^{s,q}(\Omega)} = \left[\|u\|^q_{W^{k,q}(\Omega)} + \sum_{|\alpha|=k} \int_{\Omega \times \Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^q}{|x-y|^{m+q\lambda}} \, d(x,y) \right]^{1/q}.$$

Denote by $\mathring{W}^{k,p}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

If X is a Banach space we denote by X' its dual space. If $0 < s < \infty$, denote $W^{-s,q}(\Omega) := [\mathring{W}^{s,q'}(\Omega)]'$, where $q' = q/(q-1)$.

If $\Omega \subset V \subset \overline{\Omega}$ then we denote by $L^q_{\text{loc}}(V)$ the space of all measurable functions u on Ω such that $u \in L^q(\omega)$ for each bounded open set ω with $\overline{\omega} \subset V$.

If $\Omega \subset \mathbb{R}^m$ is an open set with compact Lipschitz boundary, $0 < s < 1$, $1 < q < \infty$, denote $W^{s,q}(\partial\Omega) = \{u \in L^q(\partial\Omega); \|u\|_{W^{s,q}(\partial\Omega)} < \infty\}$ where

$$\|u\|_{W^{s,q}(\partial\Omega)} = \left[\|u\|^q_{L^q(\partial\Omega)} + \int_{\partial\Omega \times \partial\Omega} \frac{|u(x) - u(y)|^q}{|x-y|^{m-1+qs}} \, d(x,y) \right]^{1/q}.$$

Further, $W^{-s,q}(\partial\Omega) := [W^{s,q'}(\partial\Omega)]'$, where $q' = q/(q-1)$.

We denote $\mathcal{C}_c^\infty(\Omega; \mathbb{R}^m) := \{(v_1, \ldots, v_m); v_j \in \mathcal{C}_c^\infty(\Omega)\}$. Similarly for other spaces of functions.

We say that $\Omega \subset \mathbb{R}^m$ is a domain if it is an open connected set.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipchitz boundary, $-\infty < t < s < \infty$ and $1 < q < \infty$. Then the identity I is a compact mapping from $W^{s,q}(\Omega)$ to $W^{t,q}(\Omega)$.*

Proof. Suppose first that $0 \le t$. Choose r and τ such that $t < \tau < r < s$ and τ, r are not integer. Then $I : W^{s,q}(\Omega) \to W^{r,q}(\Omega)$, $I : W^{\tau,q}(\Omega) \to W^{t,q}(\Omega)$ continuously by [28, Chap. 2, §5.4, Lemma 5.4]. It is show in [37, Theorem 1.97] for Besov spaces that $I : B^{q,q}_r(\Omega) \to B^{q,q}_\tau(\Omega)$ compactly. But $W^{r,q}(\Omega) = B^{q,q}_r(\Omega)$, $W^{\tau,q}(\Omega) = B^{q,q}_\tau(\Omega)$ by [7, Theorem 6.7]. So, $I : W^{s,q}(\Omega) \to W^{t,q}(\Omega)$ compactly.

Let now $s \le 0$. Put $q' = q/(q-1)$. We have proved that $W^{-t,q'}(\Omega) \hookrightarrow W^{-s,q'}(\Omega)$ compactly. So, $[W^{-s,q'}(\Omega)]' \hookrightarrow [W^{-t,q'}(\Omega)]'$ compactly by [27, § 15, Theorem 4]. Suppose now that f_n is a bounded sequence in $W^{s,q}(\Omega)$. According to [39, Chapter IV, §1, Theorem] there exist $\tilde f_n \in [W^{-s,q'}(\Omega)]'$ such that $\tilde f_n$ are extensions of f_n and $\|\tilde f_n\| = \|f_n\|$. Since $[W^{-s,q'}(\Omega)]' \hookrightarrow [W^{-t,q'}(\Omega)]'$ compactly, there exists a sub-sequence $\tilde f_{n(k)}$ and $\tilde f \in [W^{-t,q'}(\Omega)]'$ such that $\tilde f_{n(k)} \to \tilde f$ in $[W^{-t,q'}(\Omega)]'$ as $k \to \infty$. So, $\tilde f_{n(k)} \to \tilde f$ in $W^{t,q}(\Omega)$ as $k \to \infty$. Therefore, the identity I is a compact mapping from $W^{s,q}(\Omega)$ to $W^{t,q}(\Omega)$.

If $t < 0$ and $0 \le s$, then $I : W^{s,q}(\Omega) \to L^q(\Omega)$ continuously and $I : L^q(\Omega) \to W^{t,q}(\Omega)$ compactly. If $t \le 0$ and $0 < s$, then $I : W^{s,q}(\Omega) \to L^q(\Omega)$ compactly and $I : L^q(\Omega) \to W^{t,q}(\Omega)$ continuously. In both cases $I : W^{s,q}(\Omega) \to W^{t,q}(\Omega)$ compactly. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 < p, q < \infty$ and $0 < s < \infty$. If $sp < m$ suppose moreover that $q \le mp/(m - sp)$. Then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$.*

Proof. Suppose first that $s \in \mathbb{N}$. Then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ by [19, Theorem 5.7.7].

Let now $s \notin \mathbb{N}$. Then $W^{s,p}(\Omega)$ is equal to the Besov space $B^{p,p}_s(\Omega)$ by ([7, Theorem 6.7]). If $sp > m$ then $W^{s,p}(\Omega) = B^{p,p}_s(\Omega) \hookrightarrow L^q(\Omega)$ by [1, Theorem 7.34]. If $sp \le m$ then $W^{s,p}(\Omega) = B^{p,p}_s(\Omega) \hookrightarrow L^q(\Omega)$ by [35, §46.2, Theorem]. □

3. Volume potential

Let $\lambda \ge 0$. Then there exists a unique fundamental solution $E^\lambda = (E^\lambda_{ij})$, $Q^\lambda = (Q^\lambda_j)$ of the Brinkman system

$$-\Delta \mathbf{u} + \lambda \mathbf{u} + \nabla p = 0, \quad \nabla \mathbf{u} = 0 \tag{3.1}$$

in $\mathbb{R}^m$ such that $E^\lambda(x) = o(|x|)$, $Q^\lambda(x) = o(|x|)$ as $|x| \to \infty$. (Here $\Delta f = \partial_1^2 f + \partial_2^2 f + \cdots + \partial_m^2 f$ is the Laplace operator of f.) Remember that for $i, j \in \{1, \dots, m\}$ we have

$$-\Delta E^\lambda_{ij} + \lambda E^\lambda_{ij} + \partial_i Q^\lambda_j = \delta_{ij}\delta_0, \quad \partial_1 E^\lambda_{1j} + \dots \partial_m E^\lambda_{mj} = 0,$$

$$-\Delta E^\lambda_{i,m+1} + \lambda E^\lambda_{i,m+1} + \partial_i Q^\lambda_{m+1} = 0, \quad \partial_1 E^\lambda_{1,m+1} + \dots \partial_m E^\lambda_{m,m+1} = \delta_0.$$

Clearly,

$$E^\lambda(-x) = E^\lambda(x), \quad Q^\lambda(-x) = -Q^\lambda(x).$$

If $j \in \{1, \dots, m\}$ then

$$Q^\lambda_j(x) = E^\lambda_{j,m+1}(x) = \frac{1}{\sigma_m} \frac{x_j}{|x|^m},$$

$$Q^\lambda_{m+1} = \begin{cases} \delta_0(x) + (\lambda/\sigma_m)\ln|x|^{-1}, & m = 2,\\ \delta_0(x) + (\lambda/\sigma_m)(m-2)^{-1}|x|^{2-m}, & m > 2,\end{cases}$$

where σ_m is the area of the unit sphere in $\mathbb{R}^m$. (See [38, p. 60].) The expressions of E^λ can be found in the book [38, Chapter 2]. We omit them for the sake of brevity.

For $\lambda = 0$ we obtain the fundamental solution of the Stokes system. If $i,j \in \{1,\dots,m\}$, the components of E^0 are given by

$$E^0_{ij}(x) = \frac{1}{2\sigma_m}\left\{\frac{\delta_{ij}}{(m-2)|x|^{m-2}} + \frac{x_i x_j}{|x|^m}\right\}, \quad m \geq 3 \tag{3.2}$$

$$E^0_{ij}(x) = \frac{1}{4\pi}\left\{\delta_{ij}\ln\frac{1}{|x|} + \frac{x_j x_k}{|x|^2}\right\}, \quad m = 2, \tag{3.3}$$

(see, e.g., [38, p. 16]).

If $i,j \leq m$ then

$$E^\lambda_{ij} = E^\lambda_{ji},$$

$$|E^\lambda_{ij}(x) - E^0_{ij}(x)| = O(1) \quad \text{as } |x| \to 0$$

by [38, p. 66] and

$$|\nabla E^\lambda_{ij}(x) - \nabla E^0_{ij}(x)| = O(|x|^{2-m}) \quad \text{as } |x| \to 0$$

by [21, Lemma 4.1].

If $i,j \leq m$ and $\lambda > 0$, then

$$\partial^\alpha E^\lambda_{ij}(x) = O(|x|^{-m-|\alpha|}), \quad |x| \to \infty$$

for each muliindex α. (See [18, Lemma 3.1].)

If $\mathbf{f} = (f_1,\dots,f_m)$ where $f_1,\dots,f_m$ and g are distributions in $\mathbb{R}^m$ with compact support and $\lambda \geq 0$, then

$$\mathbf{v} := E^\lambda * \begin{pmatrix}\mathbf{f}\\ g\end{pmatrix}, \qquad p := Q^\lambda * \begin{pmatrix}\mathbf{f}\\ g\end{pmatrix}$$

are well defined and

$$-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = g \quad \text{in } \mathbb{R}^m.$$

We denote $Q(x) = (Q^0_1(x),\dots,Q^0_m(x)) = (Q^\lambda_1(x),\dots,Q^\lambda_m(x))$. By $\tilde{E}^\lambda$ we denote the matrix of the type $m\times m$, where $\tilde{E}^\lambda_{ij}(x) = E^\lambda_{ij}(x)$ for $i,j \leq m$.

Proposition 3.1. *Let $\varphi,\psi \in \mathcal{C}^\infty_c(\mathbb{R}^m)$, $1 < q < \infty$ and $s \in \mathbb{R}^1$. Then there exists a constant C such that if $\mathbf{f} \in W^{s,q}(\mathbb{R}^m;\mathbb{R}^m)$ then $\varphi[Q*(\psi\mathbf{f})] \in W^{s+1,q}(\mathbb{R}^m)$ and*

$$\|\varphi[Q*(\psi\mathbf{f})]\|_{W^{s+1,q}(\mathbb{R}^m)} \leq C\|\mathbf{f}\|_{W^{s,q}(\mathbb{R}^m)}. \tag{3.4}$$

Proof. Let h_Δ be the fundamental solution of the Laplace equation given by

$$h_\Delta(x) := \begin{cases} \sigma_2^{-1}\ln|x|, & m = 2,\\ (2-m)^{-1}\sigma_m^{-1}|x|^{2-m}, & m > 2\end{cases}$$

Then $Q_j = \partial_j h_\Delta$. Thus $Q_j * (\psi f_j) = (\partial_j h_\Delta) * (\psi f_j) = \partial_j[h_\Delta * (\Psi f_j)]$. So,

$$\varphi[Q*(\psi\mathbf{f})] = \sum_{j=1}^m \{\partial_j[\varphi h_\Delta * (\Psi f_j)] - (\partial_j\varphi)[h_\Delta * (\Psi f_j)]\}.$$

[23, Proposition 3.18.5], [8, Lemma 6.36] and [13, Lemma 1.4.1.3] give that $\varphi[Q*(\psi\mathbf{f})] \in W^{s+1,q}(\mathbb{R}^m)$ and the estimate (3.4) holds. □

Proposition 3.2. *Let $0 < \lambda < \infty$, $1 < q < \infty$, $s \in \mathbb{R}^1$. Then the mapping $\mathbf{f} \mapsto \tilde{E}^\lambda * \mathbf{f}$ for $\mathbf{f} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$ can be extended by a unique way as a bounded linear operator from $W^{s,q}(\mathbb{R}^m, \mathbb{R}^m)$ to $W^{s+2,q}(\mathbb{R}^m, \mathbb{R}^m)$.*

(See [22, Proposition 6.1].)

Proposition 3.3. *Let $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, $1 < q < \infty$ and $s \in \mathbb{R}^1$. Then there exists a constant C such that if $\mathbf{f} \in W^{s,q}(\mathbb{R}^m; \mathbb{R}^m)$ then $\varphi[\tilde{E}^0 * (\psi \mathbf{f})] \in W^{s+2,q}(\mathbb{R}^m; \mathbb{R}^m)$ and*

$$\|\varphi[\tilde{E}^0 * (\psi \mathbf{f})]\|_{W^{s+2,q}(\mathbb{R}^m)} \leq C \|\mathbf{f}\|_{W^{s,q}(\mathbb{R}^m)}.$$

Proof. Let $k \in N_0$, $\mathbf{f} \in W^{k,q}(\mathbb{R}^m; \mathbb{R}^m)$. Then

$$\Delta[\tilde{E}^0 * (\psi \mathbf{f})] = \nabla[Q * (\psi \mathbf{f})] - \psi \mathbf{f} \in W^{k,q}_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^m)$$

by the definition of a fundamental solution and Proposition 3.1. Hence $\tilde{E}^0 * (\psi \mathbf{f}) \in W^{k+2,q}_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^m)$ by [23, Proposition 3.18.3 and Proposition 3.18.2]. Denote $V_{\varphi,\psi} \mathbf{f} = \varphi[\tilde{E}^\lambda * (\psi \mathbf{f})]$. Then $V_{\varphi,\psi} : W^{k,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{k+2,q}(\mathbb{R}^m; \mathbb{R}^m)$. If $\mathbf{f}_n \to \mathbf{f}$ in $W^{k,q}(\mathbb{R}^m; \mathbb{R}^m)$ and $V_{\varphi,\psi} \mathbf{f}_n \to \mathbf{g}$ in $W^{k+2,q}(\mathbb{R}^m; \mathbb{R}^m)$, then $V_{\varphi,\psi} \mathbf{f} = \mathbf{g}$ because the convolution is continuous in the sense of distributions. So, $V_{\varphi,\psi} : W^{k,q}(R^m; R^m) \to W^{k+2,q}(R^m; R^m)$ is a bounded operator by the Closed graph theorem ([30, Theorem 3.10]).

Let $k \in N_0$. Denote $q' = q/(q-1)$. Then $W^{k,q'}(\mathbb{R}^m) = \mathring{W}^{k,q'}(\mathbb{R}^m)$ by [34, §2.3.3], [35, §2.12, Theorem] and [2, Theorem 4.2.2]. Since $V_{\psi,\phi} : W_0^{k,q'}(R^m; R^m) \to W_0^{k+2,q'}(R^m; R^m)$ is bounded, the adjoint operator $[V_{\psi,\varphi}]' : W^{-k-2,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{-k,q}(\mathbb{R}^m; \mathbb{R}^m)$ is bounded, too. If $\mathbf{g}, \mathbf{h} \in \mathcal{C}_c^\infty(\mathbb{R}^m; \mathbb{R}^m)$ then

$$\int_{\mathbb{R}^m} \mathbf{g}(x) V_{\psi,\varphi} \mathbf{f}(x) \, \mathrm{d}x = \int_{\mathbb{R}^m} \mathbf{f}(y) V_{\varphi,\psi} \mathbf{g}(y) \, \mathrm{d}y,$$

because $\tilde{E}^0(-x) = \tilde{E}^0(x)$ and $\tilde{E}^0_{ij} = \tilde{E}^0_{ji}$ by (3.2) and (3.3). Thus $V_{\varphi,\psi} = [V_{\psi,\varphi}]' : W^{-k-2,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{-k,q}(\mathbb{R}^m; \mathbb{R}^m)$ is bounded.

According to According to [35, §2.4.2, Theorem 1] and [2, Theorem 4.2.2] one has

$$(L^q(\mathbb{R}^m), W^{2,q}(\mathbb{R}^m))_{1/2} = W^{1,q}(\mathbb{R}^m), \quad (W^{-2,q}(\mathbb{R}^m), L^q(\mathbb{R}^m))_{1/2} = W^{-1,q}(\mathbb{R}^m).$$

Since $V_{\varphi,\psi} : L^q(\mathbb{R}^m; \mathbb{R}^m) \to W^{2,q}(\mathbb{R}^m; \mathbb{R}^m)$, $V_{\varphi,\psi} : W^{-2,q}(\mathbb{R}^m; \mathbb{R}^m) \to L^q(\mathbb{R}^m; \mathbb{R}^m)$ are bounded, [1, p. 248] gives that $V_{\varphi,\psi} : W^{-1,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{1,q}(\mathbb{R}^m; \mathbb{R}^m)$ is bounded.

Suppose that s is not integer. Choose $k \in N$ such that $|s| < k$. Put $\theta = (s+k+2)/(2k+2)$. Then

$$(W^{-k-2,q}(\mathbb{R}^m), W^{k,q}(\mathbb{R}^m))_{\theta,q} = W^{s,q}(\mathbb{R}^m),$$

$$(W^{-k,q}(\mathbb{R}^m), W^{k+2,q}(\mathbb{R}^m))_{\theta,q} = W^{s+2,q}(\mathbb{R}^m)$$

by [7, Theorem 6.7] and [36, §2.4.2, Theorem]. Since $V_{\varphi,\psi} : W^{k,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{k+2,q}(\mathbb{R}^m; \mathbb{R}^m)$, $V_{\varphi,\psi} : W^{-k-2,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{-k,q}(\mathbb{R}^m; \mathbb{R}^m)$ are bounded operators, [32, Lemma 22.3] gives that $V_{\varphi,\psi} : W^{s,q}(\mathbb{R}^m; \mathbb{R}^m) \to W^{s+2,q}(\mathbb{R}^m; \mathbb{R}^m)$ is a bounded operator. □

4. Brinkman single layer potential

Let now $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. If $1 < q < \infty$ and $\mathbf{g} \in L^q(\partial\Omega, \mathbb{R}^m)$ then the single-layer potential for the Brinkman system $E^\lambda_\Omega \mathbf{g}$ and its associated pressure potential $Q_\Omega \mathbf{g}$ are given by

$$E^\lambda_\Omega \mathbf{g}(x) := \int_{\partial\Omega} \tilde{E}^\lambda(x-y)\mathbf{g}(y)\ \mathrm{d}\sigma(y),$$

$$Q_\Omega \mathbf{g}(x) := \int_{\partial\Omega} Q(x-y)\mathbf{g}(y)\ \mathrm{d}\sigma(y).$$

More generally, if $\mathbf{g} = (g_1, \dots, g_m)$, where g_j are distributions supported on $\partial\Omega$ then we define

$$E^\lambda_\Omega \mathbf{g}(x) := \langle \mathbf{g}, \tilde{E}^\lambda(x - \cdot)\rangle, \quad Q_\Omega \mathbf{g}(x) := \langle \mathbf{g}, Q(x - \cdot)\rangle.$$

Remark that $(E^\lambda_\Omega \mathbf{g}, Q_\Omega \mathbf{g})$ is a solution of the Brinkman system (3.1) in the set $\mathbb{R}^m \setminus \partial\Omega$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $0 < \lambda < \infty$ and $1 < q < \infty$. Then E^λ_Ω is a bounded linear operator from $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ to $W^{1,q}(\Omega;\mathbb{R}^m)$. If $\mathbf{g} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ then $Q_\Omega \mathbf{g} \in L^q_{loc}(\mathbb{R}^m)$. If Ω is bounded then E^0_Ω is a bounded linear operator from $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ to $W^{1,q}(\Omega;\mathbb{R}^m)$.*

Proof. Put $q' = q/(q-1)$. The trace operator γ_Ω is a bounded operator from $W^{1,q'}(\Omega)$ to $W^{1-1/q'}(\partial\Omega)$ by [19, Theorem 6.8.13]. For $\mathbf{g} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ define $P\mathbf{g} \in W^{-1,q}(\mathbb{R}^m;\mathbb{R}^m)$ by

$$\langle P\mathbf{g}, \mathbf{\Psi}\rangle := \langle \mathbf{g}, \gamma_\Omega \mathbf{\Psi}\rangle, \qquad \mathbf{\Psi} \in W^{1,q'}(\mathbb{R}^m;\mathbb{R}^m).$$

Since $E^\lambda_\Omega \mathbf{g} = \tilde{E}^\lambda * (P\mathbf{g})$ and $P : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to W^{-1,q}(\mathbb{R}^m;\mathbb{R}^m)$ is bounded, Proposition 3.2 gives that E^λ_Ω is a bounded linear operator from $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ to $W^{1,q}(\Omega;\mathbb{R}^m)$. Since $Q_\Omega \mathbf{g} = Q*(P\mathbf{g})$, Proposition 3.1 gives that $Q_\Omega \mathbf{g} \in L^q_{loc}(\mathbb{R}^m)$ for $\mathbf{g} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$.

Suppose now that Ω is bounded. Since $E^0_\Omega \mathbf{g} = \tilde{E}^0 * (P\mathbf{g})$, Proposition 3.3 gives that E^0_Ω is a bounded linear operator from $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ to $W^{1,q}(\Omega;\mathbb{R}^m)$. □

We denote by $\mathcal{E}^\lambda_\Omega \mathbf{g}$ the trace of $E^\lambda_\Omega \mathbf{g}$ on $\partial\Omega$.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $4/3 < q < 4$. Denote by X the set of all vector functions $\mathbf{f}$ on $\partial\Omega$ such that for each component S of $\partial\Omega$ there exists a constant c_S with $\mathbf{f} = c_S \mathbf{n}^\Omega$ on S; $Y = \{\mathbf{g} \in W^{1-1/q,q}(\partial\Omega, \mathcal{R}^2); \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{f}\mathrm{d}\sigma = 0\ \forall \mathbf{f} \in X\}$. For $\mathbf{f} = (f_1, f_2) \in W^{-1/q,q}(\partial\Omega, \mathbb{R}^2)$ and $\mathbf{c} \in \mathbb{R}^2$ denote*

$$\tilde{E}_\Omega(\mathbf{f}, \mathbf{c}) = \left[\mathcal{E}^0_\Omega \mathbf{f} + \mathbf{c}, (\langle f_1, 1\rangle_{\partial\Omega}, \langle f_2, 1\rangle_{\partial\Omega}) / \int_{\partial\Omega} 1\ \mathrm{d}\sigma\right]. \tag{4.1}$$

Then $\tilde{E}_\Omega : [W^{-1/q,q}(\partial\Omega, \mathbb{R}^2)/X] \times \mathbb{R}^2 \to Y \times \mathbb{R}^2$ is an isomorphism.

Proof. Put $s = 1 - 1/q$. Then $1/q - (s - 1/2) = 1/q - (1 - 1/q) + 1/2 = 2/q - 1/2 = (4-q)/(2q) > 0$ because $4 > q$. Further, $(s+1/2) - 1/q = 3/2 - 2/q = (3q-4)/(2q) > 0$ because $4/3 < q$. Using $W^{t,q}(\partial\Omega) = B^{q,q}_t(\partial\Omega)$ for $t \notin \mathcal{Z}$ (see for example [7, Theorem 6.7]), we get by [26, Theorem 10.5.3] that $\tilde{E}_\Omega : [W^{-1/q,q}(\partial\Omega, \mathbb{R}^2)/X] \times \mathbb{R}^2 \to Y \times \mathbb{R}^2$ is an isomorphism. □

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary and $3/2 < q < 3$. Denote by X the set of all vector functions $\mathbf{f}$ on $\partial\Omega$ such that for each component S of $\partial\Omega$ there exists a constant c_S with $\mathbf{f} = c_S \mathbf{n}^\Omega$ on S; $Y = \{\mathbf{g} \in W^{1-1/q,q}(\partial\Omega, \mathcal{R}^3); \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{f} \mathrm{d}\sigma = 0\ \forall \mathbf{f} \in X\}$. Then $\mathcal{E}^0_\Omega : W^{-1/q,q}(\partial\Omega, \mathbb{R}^3)/X \to Y$ is an isomorphism.*

Proof. Put $s = 1 - 1/q$. Then $1/q - s/2 = 1/q - [1/2 - 1/(2q)] = (3 - q)/(2q) > 0$ because $3 > q$. Further, $(s/2+1/2)-1/q = 1/2-1/(2q)+1/2-1/q = (2q-3)/(2q) > 0$ because $3/2 < q$. Using $W^{t,q}(\partial\Omega) = B^{q,q}_t(\partial\Omega)$ for $t \not\in \mathcal{Z}$ (see for example [7, Theorem 6.7]), we get by [26, Theorem 10.5.3] that $\mathcal{E}^0_\Omega : W^{-1/q,q}(\partial\Omega, \mathbb{R}^3)/X \to Y$ is an isomorphism. □

5. Boundary value problem for the Brinkman system

We begin with some auxiliary results.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary and $1 < q < \infty$. If $\mathbf{u} \in W^{1,q}(\Omega; \mathbb{R}^m)$ then*

$$\int_\Omega \nabla \cdot \mathbf{u}\ \mathrm{d}x = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}^\Omega\ \mathrm{d}\sigma. \tag{5.1}$$

Proof. If $\mathbf{u} \in \mathcal{C}^\infty(\mathbb{R}^m; \mathbb{R}^m)$ then the Green formula gives (5.1). Since $\mathcal{C}^\infty(\mathbb{R}^m)$ is a dense subset of $W^{1,q}(\Omega)$ by [1, Theorem 3.22] and the trace is a continuous operator from $W^{1,q}(\Omega)$ to $W^{1-1/q,q}(\partial\Omega)$ by [13, Theorem 1.5.1.2], we infer that (5.1) holds for $\mathbf{u} \in W^{1,q}(\Omega; \mathbb{R}^m)$. □

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Let G be a bounded component of $\mathbb{R}^m \setminus \overline{\Omega}$ and $z \in G$. Define $\mathbf{w}(x) := (x - z)/|x - z|^m$. Then $\Delta\mathbf{w} = 0$, $\nabla \cdot \mathbf{w} = 0$ in $\mathbb{R}^m \setminus \{z\}$ and*

$$\int_{\partial G} \mathbf{w} \cdot \mathbf{n}^\Omega\ \mathrm{d}\sigma = -\sigma_m$$

where $\mathbf{n}^\Omega$ denotes the unit exterior normal of Ω and σ_m is the surface of the unit sphere in $\mathbb{R}^m$.

Proof. $\mathbf{w}(x) = C_1 \nabla h(x - z)$ where C_1 is a constant and $h(x) = \ln|x|$ for $m = 2$ and $h(x) = |x|^{2-m}$ for $m > 2$. Since $\Delta h = 0$ in $\mathbb{R}^m \setminus \{0\}$, we infer that $\Delta\mathbf{w} = 0$, $\nabla \cdot \mathbf{w} = 0$ in $\mathbb{R}^m \setminus \{z\}$.

Fix $r > 0$ such that for $B := \{x; |x - z| < r\}$ we have $\overline{B} \subset G$. Since $\nabla \cdot w = 0$ in $D := G \setminus \overline{B}$, Lemma 5.1 gives

$$\int_{\partial G} \mathbf{w} \cdot \mathbf{n}^\Omega\ \mathrm{d}\sigma = -\int_{\partial D} \mathbf{w} \cdot \mathbf{n}^D\ \mathrm{d}\sigma - \int_{\partial B} \mathbf{w} \cdot \mathbf{n}^B\ \mathrm{d}\sigma = -\int_D \nabla \cdot \mathbf{w}\ \mathrm{d}x$$
$$-\int_{\partial B} \frac{x - z}{|x - z|^m} \cdot \frac{x - z}{|x - z|}\ \mathrm{d}\sigma = 0 - \int_{\partial B} |x - z|^{1-m}\ \mathrm{d}\sigma = -\sigma_m.$$

□

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary and $2 \le m \le 3$. Let $q \in (4/3, 4)$ for $m = 2$, and $q \in (3/2, 3)$ for $m = 3$. Let $\lambda = 0$. If $\mathbf{f} \in W^{-1,q}(\Omega; \mathbb{R}^m)$, $\chi \in L^q(\Omega)$ and $\mathbf{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ then there exists a solution $(\mathbf{u}, p) \in W^{1,q}(\Omega; \mathbb{R}^m) \times L^q(\Omega)$ of (1.1), (1.2) if and only if*

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega\ \mathrm{d}\sigma = \int_\Omega \chi\ \mathrm{d}x. \tag{5.2}$$

The velocity $\mathbf{u}$ *is unique and the pressure* p *is unique up to an additive constant. If*

$$\int_\Omega p \; \mathrm{d}x = 0 \tag{5.3}$$

then

$$\|\mathbf{u}\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \leq C \left(\|\mathbf{f}\|_{W^{-1,q}(\Omega)} + \|\chi\|_{L^q(\Omega)} + \|\mathbf{g}\|_{W^{1-1/q,q}(\partial\Omega)} \right) \tag{5.4}$$

where C *does not depend on* $\mathbf{f}$*,* χ *and* $\mathbf{g}$*.*

Proof. If there is a solution of (1.1), (1.2) then (5.2) holds by Lemma 5.1.

Suppose now that $(\mathbf{u}, p) \in W^{1,q}(\Omega; \mathbb{R}^m) \times L^q(\Omega)$ is a solution of (1.1), (1.2) with $\mathbf{f} \equiv 0$, $\chi \equiv 0$ and $\mathbf{g} \equiv 0$. Remember that $W^{1,q}(\Omega) = F_1^{q,2}(\Omega)$, $L^q(\Omega) = F_0^{q,2}(\Omega)$ by [37, Theorem 1.122]. Here $F_s^{q,r}(\Omega)$ denote Triebel-Lizorkin spaces. Put $s = 1 - 1/q$. If $m = 2$ then $s - 1/2 < 1/q < s + 1/2$. If $m = 3$ then $s/2 < 1/q < s/2 + 1/2$. So, [26, Theorem 10.6.2] forces that $\mathbf{u} \equiv 0$ and p is constant.

Now we prove the existence of a solution under assumption that $\mathbf{f} \equiv 0$ and $\chi \equiv 0$. Let $G(0), G(1), \dots, G(k)$ be components of $\mathrm{R}^m \setminus \overline{\Omega}$, where $G(0)$ is unbounded. Choose $z^j \in G(j)$ for $j = 1, \dots, k$. Put

$$w_j(x) = \frac{x - z^j}{|x - z^j|^m}.$$

Then $-\Delta w_j = 0$, $\nabla \cdot w^j = 0$ in $\mathbb{R}^m \setminus \{z^j\}$ by Lemma 5.2. For $\mu = (\mu_1, \dots, \mu_m) \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ put

$$V_\Omega \mu := E_\Omega^0 \mu + \sum_{j=1}^k \langle \mu, w_j \rangle w_j \qquad \text{for } m = 3,$$

$$V_\Omega \mu := E_\Omega^0 \left[\mu - \frac{(\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle)}{\sigma(\partial\Omega)} \sigma \right] + (\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle) + \sum_{j=1}^k \langle \mu, w_j \rangle w_j \quad \text{for } m = 2,$$

$$\tilde{Q}_\Omega \mu = Q_\Omega \mu \qquad \text{for } m = 3,$$

$$\tilde{Q}_\Omega \mu = Q_\Omega \left[\mu - \frac{(\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle)}{\sigma(\partial\Omega)} \sigma \right] \qquad \text{for } m = 2.$$

Here σ denotes the surface measure on $\partial\Omega$. Then $V_\Omega \mu \in W^{1,q}(\Omega; \mathbb{R}^m) \cap \mathcal{C}^\infty(\Omega; \mathbb{R}^m)$, $\tilde{Q}_\Omega \mu \in L^q(\Omega) \cap \mathcal{C}^\infty(\Omega)$ by Lemma 4.1. Moreover, $-\Delta V_\Omega \mu + \nabla \tilde{Q}_\Omega \mu = 0$, $\nabla \cdot V_\Omega \mu = 0$ in Ω. Denote by $\mathcal{V}_\Omega \mu$ the trace of $V_\Omega \mu$ on $\partial\Omega$. Proposition 4.2 and Proposition 4.3 force that $\mathcal{V}_\Omega : W^{-1/q,q}(\partial\Omega; \mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ is a Fredholm operator with index 0.

We show that the dimension of the kernel of $\mathcal{V}_\Omega$ is at most 1. Suppose that $\mu \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ and $\mathcal{V}_\Omega \mu = 0$. Since $\nabla \cdot E_\Omega^0 \nu = 0$ for all $\nu \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$, $\nabla \cdot \mathbf{d} = 0$ for all $\mathbf{d} \in \mathrm{R}^2$ and $\nabla \cdot w_j = 0$ in $G(i)$ for $j \neq i$, Lemma 5.1 gives

$$0 = \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot \mathcal{V}_\Omega \mu \; \mathrm{d}\sigma = \int_{G(i)} \nabla \cdot (\mathcal{V}_\Omega \mu - \langle \mu, w_i \rangle w_i) \; \mathrm{d}x$$

$$+ \langle \mu, w_i \rangle \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot w_i \; \mathrm{d}\sigma = \langle \mu, w_i \rangle \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot w_i \; \mathrm{d}\sigma.$$

Since

$$\int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot w_i \; \mathrm{d}\sigma \neq 0$$

by Lemma 5.2, we infer that

$$\langle \mu, w_i \rangle = 0 \qquad \text{for } i = 1, \dots, k. \tag{5.5}$$

We now show that there exist constants $c_0, c_1, \dots, c_k$ such that

$$\mu = c_j \mathbf{n}^\Omega \sigma \qquad \text{on } \partial G(j). \tag{5.6}$$

If $m = 3$ then Proposition 4.3 gives that there exist constants $c_0, c_1, \dots, c_k$ such that (5.6) holds. Let now $m = 2$. Then $0 = \mathcal{V}_\Omega \mu = \mathcal{E}^0_\Omega \tilde{\mu} + (\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle)$, where

$$\tilde{\mu} = \mu - \frac{(\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle)}{\sigma(\partial\Omega)} \sigma.$$

Let $\tilde{E}_\Omega$ be given by (4.1). Since

$$\tilde{E}_\Omega(\tilde{\mu}, (\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle)) = [\mathcal{V}_\Omega \mu, 0] = [0, 0]$$

Proposition 4.2 gives that $(\langle \mu_1, 1 \rangle, \langle \mu_2, 1 \rangle) = (0, 0)$ and there are constants $c_0, \dots, c_k$ such that $\tilde{\mu} = c_j \mathbf{n}^\Omega$ on $\partial G(j)$. So, $\mu = \tilde{\mu} = c_j \mathbf{n}^\Omega$ on $\partial G(j)$ for $j = 0, \dots, k$. Therefore (5.6) holds for $m = 2, 3$. If $i \geq 1$ then (5.5), (5.6) give

$$\begin{aligned} 0 = \langle \mu, w_i \rangle &= \sum_{j=0}^{k} \int_{\partial G(j)} c_j \mathbf{n}^\Omega \cdot w_i \, \mathrm{d}\sigma = - \sum_{j \neq 0, i} c_j \int_{G(j)} \nabla \cdot w_i \, \mathrm{d}x \\ &+ c_i \int_{\partial G(i)} \mathbf{n}^\Omega \cdot w_i \, \mathrm{d}\sigma + c_0 \int_{\partial G(0)} \mathbf{n}^\Omega \cdot w_i \, \mathrm{d}\sigma \\ &= -c_i \int_{\partial G(i)} \sigma_m + c_0 \int_{\partial G(0)} \mathbf{n}^\Omega \cdot w_i \, \mathrm{d}\sigma \end{aligned}$$

by Lemma 5.1 and Lemma 5.2. Therefore

$$c_i = c_0 \sigma_m^{-1} \int_{\partial G(0)} \mathbf{n}^\Omega \cdot w_i \, \mathrm{d}\sigma.$$

So, the dimension of the kernel of $\mathcal{V}_\Omega$ is at most 1.

Since $\mathcal{V}_\Omega : W^{-1/q,q}(\partial\Omega; \mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ is a Fredholm operator with index 0, the co-dimension of the range of $\mathcal{V}_\Omega$ is at most 1. Since $-\Delta V_\Omega \mu + \nabla \tilde{Q}_\Omega \mu = 0$, $\nabla \cdot V_\Omega \mu = 0$ in Ω, the condition (5.2) gives that

$$\mathcal{V}_\Omega(W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)) = \{\mathbf{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m); \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma = 0\}.$$

So, if $\mathbf{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ satisfies

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma = 0,$$

then there exists $\mu \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ such that $(V_\Omega \mu, \tilde{Q}_\Omega \mu) \in W^{1,q}(\Omega; \mathbb{R}^m) \times L^q(\Omega)$ is a solution of (1.1), (1.2) with $\mathbf{f} \equiv 0$, $\chi \equiv 0$.

Let $\mathbf{f} \in W^{-1,q}(\Omega; \mathbb{R}^m)$, $\chi \in L^q(\Omega)$ and $\mathbf{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ satisfy (5.2). Choose an open ball B in $\mathbb{R}^m$ such that $\overline{\Omega} \subset B$. Put $\tilde{\chi} := \chi$ in Ω, $\tilde{\chi} := d$ in $\mathbb{R}^m \setminus \Omega$, where d is a constant such that

$$\int_B \tilde{\chi} \, \mathrm{d}x = 0. \tag{5.7}$$

Denote $X := \{\mathbf{v} \in \mathring{W}^{1,q/(q-1)}(B; \mathbb{R}^m); \mathbf{v} = 0 \text{ in } B \setminus \Omega\}$. Then $\mathring{W}^{1,q/(q-1)}(\Omega; \mathbb{R}^m) = \{\mathbf{v}|_\Omega; \mathbf{v} \in X\}$ by [2, Theorem 9.1.3] and thus $\mathbf{f}$ is a bounded linear operator on X. According to Hahn-Banach theorem ([33, Theorem 4.3-A]) there exists $\tilde{\mathbf{f}} \in$

$W^{-1,q}(B;\mathbb{R}^m)$ such that $\langle \tilde{\mathbf{f}}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle$ for all $\mathbf{v} \in X$. Since (5.7) holds there exists a solution $(\tilde{\mathbf{u}}, \tilde{p}) \in W^{1,q}(B,\mathbb{R}^m) \times L^q(B)$ of

$$-\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} = \tilde{\mathbf{f}}, \quad \nabla\cdot\tilde{\mathbf{v}} = \tilde{\chi} \qquad \text{in } B,$$

$$\tilde{\mathbf{u}} = 0 \qquad \text{on } \partial B.$$

(See [11, Theorem 2.1].) Then $-\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} = \mathbf{f}$, $\nabla\cdot\tilde{\mathbf{u}} = \chi$ in Ω. Lemma 5.1 forces

$$\int_{\partial\Omega} \tilde{\mathbf{u}}\cdot\mathbf{n}^\Omega \, d\sigma = \int_\Omega \nabla\cdot\tilde{\mathbf{u}} \, dx = \int_\Omega \chi \, dx.$$

Put $\tilde{\mathbf{g}} = \mathbf{g} - \tilde{\mathbf{u}}$ on $\partial\Omega$. Then $\tilde{\mathbf{g}} \in W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ by [19, Theorem 6.8.13]. According to (5.2) we have

$$\int_{\partial\Omega} \tilde{\mathbf{g}}\cdot\mathbf{n}^\Omega \, d\sigma = \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}^\Omega \, d\sigma - \int_{\partial\Omega} \tilde{\mathbf{u}}\cdot\mathbf{n}^\Omega \, d\sigma = \int_\Omega \chi \, dx - \int_\Omega \chi \, dx = 0.$$

We have proved that there exists a solution $(\mathbf{v}, \rho) \in W^{1,q}(\Omega,\mathbb{R}^m) \times L^q(\Omega)$ of

$$-\Delta\mathbf{v} + \nabla\rho = 0, \quad \nabla\cdot\mathbf{v} = 0 \qquad \text{in } \Omega,$$

$$\mathbf{v} = \tilde{\mathbf{g}} \qquad \text{on } \partial\Omega.$$

Put $\mathbf{u} := \tilde{\mathbf{u}} + \mathbf{v}$, $p := \tilde{p} + \rho$. Then $(\mathbf{u}, p) \in W^{1,q}(\Omega;\mathbb{R}^m) \times L^q(\Omega)$ is a solution of (1.1), (1.2).

Define

$$L(\mathbf{u}, p) := (-\Delta\mathbf{u} + \nabla p, \nabla\cdot\mathbf{p}, \mathbf{u}|_{\partial\Omega}).$$

Then L is a bounded linear operator from $W^{1,q}(\Omega;\mathbb{R}^m)\times L^q(\Omega)$ to $W^{-1,q}(\Omega;\mathbb{R}^m)\times L^q(\Omega) \times W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$. (See [19, Theorem 6.8.13], [37, Theorem 1.122], [25, Proposition 7.6].) Denote by Y the set of $(\mathbf{u}, p)$ from $W^{1,q}(\Omega;\mathbb{R}^m) \times L^q(\Omega)$ satisfying (5.3). Further denote by Z the set of all $(\mathbf{f}, \chi, \mathbf{g})$ from $(W^{-1,q}(\Omega;\mathbb{R}^m) \times L^q(\Omega) \times W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m))$ satisfying (5.2). We have proved that $L : Y \to Z$ is an isomorphism. So, $L^{-1} : Z \to Y$ is an isomorphism, too. Thus there exists a constant C such that (5.4) holds. □

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 \le s < \infty$, $1 < q < \infty$ and $0 \le \lambda, \beta < \infty$. Suppose that one of the following conditions is fulfilled:*

(1) $s = 1$ and $q = 2$.
(2) $\Omega \subset \mathbb{R}^2$, $s = 1$ and $4/3 < q < 4$.
(3) $\Omega \subset \mathbb{R}^3$, $s = 1$ and $3/2 < q < 3$.
(4) $\partial\Omega$ is of class $\mathcal{C}^1$ and $s = 1$.
(5) $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ with $k \in N$ and $s \le k + 1$.

If $\mathbf{f} \in W^{s-2,q}(\Omega;\mathbb{R}^m)$, $\chi \in W^{s-1,q}(\Omega)$ and $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega;\mathbb{R}^m)$ then there exists a solution $(\mathbf{u}, p) \in W^{s,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega)$ of (1.1), (1.4) if and only if (5.2) holds. The velocity $\mathbf{u}$ is unique and the pressure p is unique up to an additive constant. If p satisfies (5.3) then

$$\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|p\|_{W^{s-1,q}(\Omega)} \le C\left(\|\mathbf{f}\|_{W^{s-2,q}(\Omega)} + \|\chi\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{s-1/q,q}(\partial\Omega)}\right)$$

where C does not depend on $\mathbf{f}$, χ and $\mathbf{g}$.

Proof. Lemma 5.1 forces that (5.2) is a necessary condition for the solvability of the problem (1.1), (1.4).

Suppose first that $\beta = 0$. Put $X_{s,q} := W^{s,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega)$, $Y_{s,q} := W^{s-2,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega) \times W^{s-1/q,q}(\partial\Omega;\mathbb{R}^m)$. For $\mu \in \mathbb{R}^1$ define

$$B_\mu(\mathbf{u},p) := (-\Delta\mathbf{u} + \mu\mathbf{u} + \nabla p, \nabla\cdot\mathbf{u}, \gamma_\Omega\mathbf{u}),$$

where γ_Ω is the trace operator. Then B_μ is a bounded linear operator from $X_{s,q}$ to $Y_{s,q}$ by [13, Theorem 1.4.4.6] and [13, Theorem 1.5.1.2]. Since $B_\lambda(\mathbf{u},p) - B_0(\mathbf{u},p) = (\lambda\mathbf{u},0,0)$, the operator $B_\lambda - B_0 : X_{s,q} \to Y_{s,q}$ is compact by Proposition 2.1. So, $B_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0 if and only if $B_0 : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0.

Denote by *Ker* B_λ the kernel of B_λ. If dim *Ker* $B_\lambda \le 1$ then Ker $B_\lambda = \{(\mathbf{u},p); \mathbf{u} \equiv 0, p \text{ is constant}\}$. Suppose now that $B_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0 and dim *Ker* $B_\lambda \le 1$. Then the co-dimension of the range of B_λ is equal to 1. So, (5.2) is a necessary and sufficient condition for the solvability of the problem (1.1), (1.2). Denote by Z the space of all $p \in W^{s-1,q}(\Omega)$ satisfying (5.3), by W the space of $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega;\mathbb{R}^m)$ satisfying (5.2), $X := W^{s,q}(\Omega;\mathbb{R}^m) \times Z$ and $Y := W^{s-2,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega) \times W$. Then B_λ is an isomorphism X onto Y. So, propositions of the theorem hold.

Let $s = 1$ and $q = 2$. If $(\mathbf{u},p) \in W^{1,2}(\Omega;\mathbb{R}^m) \times L^2(\Omega)$ is a solution of (1.1), (1.2) with $\mathbf{f} \equiv 0$, $\chi \equiv 0$ and $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$ and p is constant by [6, Theorem IV.8.1]. Moreover $B_0 : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0 by [6, Theorem IV.5.2]. Thus $B_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0 and (5.2) is a necessary and sufficient condition for the solvability of the problem (1.1), (1.2).

If $\partial\Omega$ is of class $\mathcal{C}^1$ and $s = 1$ then $B_0 : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0 by [11, Theorem 2.1]. So, $B_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0. If $q \ge 2$ then $X_{s,q} \hookrightarrow X_{1,2}$, $Y_{s,q} \hookrightarrow Y_{1,2}$ by Hölder's inequality and $X_{s,q}$ is a dense subset of $X_{1,2}$, $Y_{s,q}$ is a dense subset of $Y_{1,2}$ by [1, Theorem 3.22]. If $q \le 2$ then $X_{1,2} \hookrightarrow X_{s,q}$, $Y_{1,2} \hookrightarrow Y_{s,q}$ by Hölder's inequality and $X_{1,2}$ is a dense subset of $X_{s,q}$, $Y_{1,2}$ is a dense subset of $Y_{s,q}$ by [1, Theorem 3.22]. So, [23, Lemma 1.8.4] gives that the kernel of $B_\lambda : X_{s,q} \to Y_{s,q}$ is the same as the kernel of $B_\lambda : X_{1,2} \to Y_{1,2}$. Hence the dimension of the kernel of $B_\lambda : X_{s,q} \to Y_{s,q}$ is equal to 1. We have proved that the proposition of the Theorem is true.

Suppose now that $s = 1$ and $2 \le m \le 3$. If $m = 2$ suppose that $4/3 < q < 4$. If $m = 3$ suppose that $3/2 < q < 3$. Then $B_0 : X_{1,q} \to Y_{1,q}$ is a Fredholm operator with index 0 by Proposition 5.3. So, $B_\lambda : X_{1,q} \to Y_{1,q}$ is a Fredholm operator with index 0. If $q \ge 2$ then $X_{1,q} \hookrightarrow X_{1,2}$, $Y_{1,q} \hookrightarrow Y_{1,2}$ by Hölder's inequality and $X_{1,q}$ is a dense subset of $X_{1,2}$, $Y_{1,q}$ is a dense subset of $Y_{1,2}$ by [1, Theorem 3.22]. If $q \le 2$ then $X_{1,2} \hookrightarrow X_{s,q}$, $Y_{1,2} \hookrightarrow Y_{1,q}$ by Hölder's inequality and $X_{1,2}$ is a dense subset of $X_{1,q}$, $Y_{1,2}$ is a dense subset of $Y_{1,q}$ by [1, Theorem 3.22]. So, [23, Lemma 1.8.4] gives that the kernel of $B_\lambda : X_{1,q} \to Y_{1,q}$ is the same as the kernel of $B_\lambda : X_{1,2} \to Y_{1,2}$. Hence the dimension of the kernel of $B_\lambda : X_{1,q} \to Y_{1,q}$ is equal to 1. We have proved that the proposition of the Theorem is true.

Suppose now that $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ with $k \in N$ and $s = k+1$. Then $B_0 : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0 by [5, Theorem 4.8]. So, $B_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0. Since the kernel of $B_\lambda : X_{s,q} \to Y_{s,q}$ is a subset of the kernel of $B_\lambda : X_{1,q} \to Y_{1,q}$, the dimension of the kernel of $B_\lambda : X_{s,q} \to Y_{s,q}$ is at most 1. We have proved that the proposition of the Theorem is true.

Suppose now that $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ with $k \in N$ and $k < s < k+1$. Define

$$\tilde{B}_\mu(\mathbf{u}, p) := (-\Delta\mathbf{u} + \mu\mathbf{u} + \nabla p, \nabla\cdot\mathbf{u} + \int_\Omega p \, \mathrm{d}x, \gamma_\Omega\mathbf{u}).$$

Since $B_\lambda : X_{k,q} \to Y_{k,q}$ and $B_\lambda : X_{k+1,q} \to Y_{k+1,q}$ are Fredholm operators with index 0, and the operator $\tilde{B}_\lambda - B_\lambda$ is finite-dimensional, $\tilde{B}_\lambda : X_{k,q} \to Y_{k,q}$ and $\tilde{B}_\lambda : X_{k+1,q} \to Y_{k+1,q}$ are Fredholm operators with index 0. Suppose now that $(\mathbf{u}, p) \in X_{k,q}$ and $\tilde{B}_\lambda(\mathbf{u}, p) = 0$. According to Green's formula

$$0 = \int_\Omega \left(\nabla\cdot\mathbf{u} + \int_\Omega p \, \mathrm{d}x\right) \mathrm{d}x = \int_{\partial\Omega} \mathbf{u}\cdot\mathbf{n}^\Omega \, \mathrm{d}\sigma + \int_\Omega p \, \mathrm{d}x \cdot \int_\Omega 1 \, \mathrm{d}x = \int_\Omega p \, \mathrm{d}x \cdot \int_\Omega 1 \, \mathrm{d}x.$$

Since $\int_\Omega p \, \mathrm{d}x = 0$ we have $B_\lambda(\mathbf{u}, p) = 0$. We have proved that $(\mathbf{u}, p) = 0$. Hence $\tilde{B}_\lambda : X_{k,q} \to Y_{k,q}$ and $\tilde{B}_\lambda : X_{k+1,q} \to Y_{k+1,q}$ are isomorphisms. We now use the real interpolation. Choose $\theta \in (0, 1)$ such that $s = (1-\theta)k + \theta k$. Then

$$(X_{k,q}, X_{k+1,q})_{\theta,q} = X_{s,q}, \quad (Y_{k,q}, Y_{k+1,q})_{\theta,q} = Y_{s,q}$$

by [7, Corollary 6.8] and [32, Lemma 41.3]. So, [3, Theorem 13.7.1] forces that $\tilde{B}_\lambda : X_{s,q} \to Y_{s,q}$ is an isomorphism, too. (We can also use the complex interpolation and [16, Proposition 2.4], [35, §1.11.3], [3, Theorem 13.7.1].) Therefore $B_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0. Since the kernel of $B_\lambda : X_{s,q} \to Y_{s,q}$ is a subset of the kernel $B_\lambda : X_{k,q} \to Y_{k,q}$, the dimension of the kernel of $B_\lambda : X_{s,q} \to Y_{s,q}$ is equal to 1. We have proved that propositions of the Theorem hold.

Suppose now that $\beta > 0$. Define

$$C_\lambda(\mathbf{u}, p) := (-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p, \nabla\cdot\mathbf{u}, \gamma_\Omega\mathbf{u} + \beta\int_\Omega \mathbf{u} \, \mathrm{d}x).$$

Since the operator $C_\lambda - B_\lambda$ is finite-dimensional, the operator $C_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0. Let now $(\mathbf{u}, p) \in X_{s,q}$ be such that $C_\lambda(\mathbf{u}, p) = 0$. Then

$$-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0, \quad \nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega,$$
$$\mathbf{u} = -\beta\int_\Omega \mathbf{u} \, \mathrm{d}x \quad \text{on } \partial\Omega.$$

Thus there is a constant c such that $\mathbf{u} \equiv -\beta\int_\Omega \mathbf{u} \, \mathrm{d}x$, $p \equiv c$. Therefore

$$0 = \int_\Omega (\mathbf{u} + \beta\int_\Omega \mathbf{u} \, \mathrm{d}x) \, \mathrm{d}x = \int_\Omega \mathbf{u} \, \mathrm{d}x(1 + \beta\int_\Omega 1 \, \mathrm{d}x).$$

Since $\beta > 0$ we infer that $\int_\Omega \mathbf{u} \, \mathrm{d}x = 0$. Hence $B_\lambda(\mathbf{u}, p) = 0$. We have proved that $\mathbf{u} \equiv 0$. Since the dimension of the kernel of $C_\lambda : X_{s,q} \to Y_{s,q}$ is equal to 1 and the the operator $C_\lambda : X_{s,q} \to Y_{s,q}$ is a Fredholm operator with index 0, the co-dimension of the range of $C_\lambda : X_{s,q} \to Y_{s,q}$ is equal to 1. Therefore (5.2) is a necessary and sufficient condition for the solvability of the problem (1.1), (1.4). So, C_λ is an isomorphism X onto Y. □

6. Darcy-Forchheimer-Brinkman system

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $k \in \mathbb{N}$ and $1 < q < \infty$. Then there is a constant C such that the following holds: If $\mathbf{w} \in W^{1,q}(\Omega; \mathbb{R}^k)$ then $|\mathbf{w}| \in W^{1,q}(\Omega)$ and*

$$\| \, |\mathbf{w}| \, \|_{W^{1,q}(\Omega)} \le C\|\mathbf{w}\|_{W^{1,q}(\Omega)}.$$

Proof. Fix $\mathbf{w} \in W^{1,q}(\Omega;\mathbb{R}^k)$. Put $g_i := |w_i|$. Then $g_i \in W^{1,q}(\Omega)$ and $\|g_i\|_{W^{1,q}(\Omega)} = \|w_i\|_{W^{1,q}(\Omega)}$ by [20, Theorem 6.17]. For $\epsilon \geq 0$ put $g^\epsilon := |(g_1+\epsilon,\dots,g_k+\epsilon)|$. Remark that $g^0 = |\mathbf{w}|$,

$$\|g^0\|_{L^q(\Omega)} \leq k\|\mathbf{w}\|_{L^q(\Omega)}$$

and $g^\epsilon \to g^0$ in $L^q(\Omega)$ as $\epsilon \to 0_+$. If $\epsilon > 0$ then

$$\partial_j g^\epsilon(x) = \frac{1}{2g^\epsilon(x)} \sum_{i=1}^m (g_i(x)+\epsilon)\partial_j g_i(x).$$

So,

$$|\partial_j g^\epsilon| \leq |\partial_j g_1,\dots,\partial_j g_k)| \leq |(\partial_j g_1| + \cdots + |\partial_j g_k|.$$

Therefore

$$\|\partial_j g^\epsilon\|_{L^q(\Omega)} \leq \sum_{i=1}^k \|\partial_j g_i\|_{L^q(\Omega)}.$$

If $g^0(x) > 0$ then $\partial_j g^\epsilon(x) \to \frac{1}{2}|g^0(x)|^{-1}\sum_{i=1}^m g_i(x)\partial_j g_i(x)$ as $\epsilon \to 0_+$. If $g^0(x) = 0$ then $\partial_j g_i(x) = 0$ by [20, Theorem 6.17] and thus $\partial_j g^\epsilon(x) = 0$. Put

$$f(x) := \frac{1}{2g^0(x)} \sum_{i=1}^m g_i(x)\partial_j g_i(x) \quad \text{for } g^0(x) > 0,$$

$$f(x) := 0 \qquad \text{for } g^0(x) = 0.$$

Then $\partial_j g^\epsilon \to f$ in $L^q(\Omega)$ as $\epsilon \to 0_+$ by Lebesgue's theorem. (See [4, Theorem 3.12].) So, $g^{1/n}$ is a Cauchy sequence in $W^{1,q}(\Omega)$. Therefore there exists $h \in W^{1,q}(\Omega)$ such that $g^{1/n} \to h$ in $W^{1,q}(\Omega)$. Since $g^{1/n} \to |\mathbf{w}|$ in $L^q(\Omega)$, we infer that $h = |\mathbf{w}|$. Since

$$\|g^{1/n}\|_{W^{1,q}(\Omega)} \leq k^2\|\mathbf{w}\|_{W^{1,q}(\Omega)},$$

we infer that

$$\|\,|\mathbf{w}|\,\|_{W^{1,q}(\Omega)} \leq k^2\|\mathbf{w}\|_{W^{1,q}(\Omega)}.$$

□

Remark 6.2. Clearly $\|\,|\mathbf{w}|-|\mathbf{v}|\,\|_{L^r(\Omega)} \leq \|\mathbf{w}-\mathbf{v}\|_{L^r(\Omega)}$. But in general, it does not exist a constant C such that

$$\|\,|\mathbf{w}|-|\mathbf{v}|\,\|_{W^{1,q}(\Omega)} \leq \|\mathbf{w}-\mathbf{v}\|_{W^{1,q}(\Omega)}.$$

This shows the following easy example: Let $I = (0,1)$. Fix $q \in (1,\infty)$. Put $f_\alpha(t) := t^\alpha$, $g_\alpha(t) := t^\alpha - 1$. Then $f'_\alpha(t) = g'_\alpha(t) = \alpha t^{\alpha-1}$. So, $f_\alpha, g_\alpha \in W^{1,q}(I)$ if and only if $\alpha > (q-1)/q$. Since $f_\alpha - g_\alpha \equiv 1$, we have $\|f_\alpha - g_\alpha\|_{W^{1,q}(I)} = 1$. Since $|f_\alpha| - |g_\alpha| = 2t^\alpha - 1$, we have $\partial_t(|f_\alpha(t)| - |g_\alpha(t)|) = 2\alpha t^{\alpha-1}$. So,

$$\int_0^1 |\partial_t(|f_\alpha(t)| - |g_\alpha(t)|)|^q \,\mathrm{d}t = (2\alpha)^q \int_0^1 t^{q\alpha-q}\,\mathrm{d}t = \frac{(2\alpha)^q}{\alpha q - q + 1}.$$

If $\alpha \searrow (q-1)/q$ then $\|\,|f_\alpha| - |g_\alpha|\,\|^q_{W^{1,q}(I)} \geq \frac{(2\alpha)^q}{\alpha q - q+1} \to \infty$.

Lemma 6.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 \leq s < 3$ and $\max(1, m/3) < q < \infty$. For $\mathbf{u},\mathbf{v} \in W^{s,q}(\Omega;\mathbb{R}^m)$ define*

$$A(\mathbf{u},\mathbf{v}) := |\mathbf{u}|\mathbf{v}.$$

(1) Then there is a positive constant C such that the following holds: If $\mathbf{u},\mathbf{v} \in W^{s,q}(\Omega;\mathbb{R}^m)$ then $A(\mathbf{u},\mathbf{v}) \in W^{s-2,q}(\Omega;\mathbb{R}^m)$ and

$$\|A(\mathbf{u},\mathbf{v})\|_{W^{s-2,q}(\Omega)} \leq C\|\mathbf{u}\|_{W^{s,q}(\Omega)}\|\mathbf{v}\|_{W^{s,q}(\Omega)}.$$

(2) Suppose that $s \le 2$*. If* $s < 2$ *and* $sq < m = 3$ *suppose moreover that* $q \ge 6/(3+2s)$*. If* $s < 2$ *and* $m/(m-2+s) < q < m/s$ *suppose moreover that* $q \ge m/(2+s)$*. Then*

$$\|A(\mathbf{u},\mathbf{u}) - A(\mathbf{v},\mathbf{v})\|_{W^{s-2,q}(\Omega)} \le C\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)} \left(\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|\mathbf{v}\|_{W^{s,q}(\Omega)}\right). \tag{6.1}$$

Proof. According to Lemma 6.1 there exists a constant C_1 such that

$$\| \; |\mathbf{w}| \; \|_{W^{1,q}(\Omega)} \le C_1 \|\mathbf{w}\|_{W^{s,q}(\Omega)}$$

for all $\mathbf{w} \in W^{s,q}(\Omega;\mathbb{R}^m)$. Since $\min(s,1) > s-2$ and $s+1-(s-2) = 3 > m/q$, Lemma 7.1 forces that there is a constant C_2 such that

$$\|fg\|_{W^{s-2,q}(\Omega)} \le C_2 \|f\|_{W^{1,q}(\Omega)} \|g\|_{W^{s,q}(\Omega)}$$

for all $f \in W^{1,q}(\Omega)$ and $g \in W^{s,q}(\Omega)$. If $\mathbf{u},\mathbf{v} \in W^{s,q}(\Omega;\mathbb{R}^m)$ then

$$\|A(\mathbf{u},\mathbf{v})\|_{W^{s-2,q}(\Omega)} \le C_2 m \| \; |\mathbf{u}| \; \|_{W^{1,q}(\Omega)} \|\mathbf{v}\|_{W^{s,q}(\Omega)} \le C_1 C_2 m \|\mathbf{u}\|_{W^{s,q}(\Omega)} \|\mathbf{v}\|_{W^{s,q}(\Omega)}.$$

As Remark 6.2 shows, the proof of the second part of Lemma will be a bit complicated. Since $A(\mathbf{u},\mathbf{u}) - A(\mathbf{v},\mathbf{v}) = A(\mathbf{u},\mathbf{u}-\mathbf{v}) + (|\mathbf{u}| - |\mathbf{v}|)\mathbf{v}$, we have

$$\begin{aligned} &\|A(\mathbf{u},\mathbf{u}) - A(\mathbf{v},\mathbf{v})\|_{W^{s-2,q}(\Omega)} \\ &\le C_1 C_2 m \|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)} \|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|(|\mathbf{u}| - |\mathbf{v}|)\mathbf{v}\|_{W^{s-2,q}(\Omega)}. \end{aligned} \tag{6.2}$$

Suppose that $s \le 2$. Suppose first that $sq \ge m$. According to Lemma 2.2 there is a positive constant C_3 such that

$$\|f\|_{L^{2q}(\Omega)} \le C_3 \|f\|_{W^{s,q}(\Omega)} \qquad \forall f \in W^{s,q}(\Omega).$$

If $\mathbf{u},\mathbf{v} \in W^{s,q}(\Omega;\mathbb{R}^m)$ then

$$\begin{aligned} \|(|\mathbf{u}| - |\mathbf{v}|)\mathbf{v}\|_{L^q(\Omega)} &\le \||\mathbf{u}-\mathbf{v}|\mathbf{v}\|_{L^q(\Omega)} \le \| \; |\mathbf{u}-\mathbf{v}| \; \|_{L^{2q}(\Omega)} \|\mathbf{v}\|_{L^{2q}(\Omega)} \\ &\le C_3^2 m^2 \|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)} \|\mathbf{v}\|_{W^{s,q}(\Omega)} \end{aligned}$$

by Hölder's inequality. So,

$$\|(|\mathbf{u}| - |\mathbf{v}|)\mathbf{v}\|_{W^{s-2,q}(\Omega)} \le C_3^2 m^2 \|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)} \|\mathbf{v}\|_{W^{s,q}(\Omega)}.$$

This and (6.2) force that (6.1) holds with $C \ge C_1 C_2 m + C_3^2 m^2$.

Suppose now that $s \le 2$ and $sq < m$. Put $r = mq/(m-sq)$. Then there is a constant C_4 such that

$$\|f\|_{L^r(\Omega)} \le C_4 \|f\|_{W^{s,q}(\Omega)} \qquad \text{for } f \in W^{s,q}(\Omega) \tag{6.3}$$

by Lemma 2.2. We show that $r/2 \ge 1$. Since $q > m/3$ we have for $m \ge 4$ that $r/2 > m(m/3)/[2(m-m/3)] = m/4 \ge 1$. If $m = 2$ then $r/2 = q/(2-sq) \ge q/(2-1) = q > 1$. Suppose now that $m = 3$. Since $q \ge 6/(3+2s)$ we obtain

$$\frac{r}{2} = \frac{3q}{2(3-sq)} \ge \frac{18/(3+2s)}{6 - 12s/(3+2s)} = \frac{3}{(3+2s)-2s} = 1.$$

Hölder's inequality forces

$$\|(|\mathbf{u}| - |\mathbf{v}|)\mathbf{v}\|_{L^{r/2}(\Omega)} \le \|\mathbf{u}-\mathbf{v}\|_{L^r(\Omega)} \|\mathbf{v}\|_{L^r(\Omega)}.$$

Using (6.3)

$$\|(|\mathbf{u}| - |\mathbf{v}|)\mathbf{v}\|_{L^{r/2}(\Omega)} \le C_4^2 m^2 \|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)} \|\mathbf{v}\|_{W^{s,q}(\Omega)}. \tag{6.4}$$

Suppose first that $s = 2$. Since $m/3 < q$ we have $r/2 = qm/(2m-4q) > qm/(2m - 4m/3) = q \cdot 3/2 > q$. So, there is a constant C_5 such that

$$\|f\|_{L^q(\Omega)} \le C_5 \|f\|_{L^{r/2}(\Omega)} \qquad \forall f \in L^{r/2}(\Omega). \tag{6.5}$$

According to (6.4) we obtain

$$\|(|\mathbf{u}|-|\mathbf{v}|)\mathbf{v}\|_{W^{s-2,q}(\Omega)} = \|(|\mathbf{u}|-|\mathbf{v}|)\mathbf{v}\|_{L^q(\Omega)} \le C_5\|(|\mathbf{u}|-|\mathbf{v}|)\mathbf{v}\|_{L^{r/2}(\Omega)}$$
$$\le C_5C_4^2m^2\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)}\|\mathbf{v}\|_{W^{s,q}(\Omega)}.$$

Therefore (6.2) gives that (6.1) holds with $C \ge C_1C_2m + C_5C_4m^2$.

Let now $s<2$ and $sq<m$. If $m=2$ then $r/2 \ge q$ as we have proved. So, there is a constant C_5 such that (6.5) holds. Therefore there is a constant C_6 such that

$$\|f\|_{W^{s-2,q}(\Omega)} \le C_6\|f\|_{L^{r/2}(\Omega)} \qquad \forall f \in L^{r/2}(\Omega). \tag{6.6}$$

Suppose now that $m \ge 3$. Put $q' = q/(q-1)$ and $t = (r/2)/(r/2-1)$. Suppose first that $(2-s)q' \ge m$. According to Lemma 2.2 there is a constant C_7 such that

$$\|g\|_{L^t(\Omega)} \le C_7\|g\|_{W^{2-s,q'}(\Omega)} \qquad \forall g \in W^{2-s,q'}(\Omega).$$

If $f \in L^{r/2}(\Omega)$ and $g \in W^{2-s,q'}(\Omega)$ then Hölder's inequality yields

$$\left|\int_\Omega fg \,\mathrm{d}x\right| \le \|f\|_{L^{r/2}(\Omega)}\|g\|_{L^t(\Omega)} \le \|f\|_{L^{r/2}(\Omega)}C_7\|g\|_{W^{2-s,q'}(\Omega)}.$$

Thus $f \in W^{s-2,q}(\Omega)$ and (6.6) holds with $C_6 \ge C_7$. Suppose now that $(2-s)q' < m$. Put $\tau = mq'/[m-(2-s)q']$. According to Lemma 2.2 there is a constant C_8 such that

$$\|g\|_{L^\tau(\Omega)} \le C_8\|g\|_{W^{2-s,q'}(\Omega)} \qquad \forall g \in W^{2-s,q'}(\Omega). \tag{6.7}$$

Clearly,

$$t = \frac{r/2}{r/2-1} = \frac{(mq)/(2m-2sq)}{(mq)/(2m-2sq)-1} = \frac{mq}{mq-2m+2sq},$$
$$\tau = \frac{mq'}{m-(2-s)q'} = \frac{mq/(q-1)}{m-(2-s)q/(q-1)} = \frac{mq}{mq-m-(2-s)q}.$$

Thus $\tau \ge t$ if and only if $m+(2-s)q \ge 2m-2sq$, i.e. if $(2+s)q \ge m$. Since $q' < m/(2-s)$ we have

$$q = \frac{q'}{q'-1} > \frac{m/(2-s)}{m/(2-s)-1} = \frac{m}{m-2+s}.$$

So, $q \ge m/(2+s)$ by assumptions. Therefore $\tau \ge t$. Thus there is a constant C_9 such that

$$\|g\|_{L^t(\Omega)} \le C_9\|g\|_{L^\tau(\Omega)} \qquad \forall g \in L^\tau(\Omega).$$

If $f \in L^{r/2}(\Omega)$ and $g \in W^{2-s,q'}(\Omega)$ then Hölder's inequality and (6.7) yield

$$\left|\int_\Omega fg \,\mathrm{d}x\right| \le \|f\|_{L^{r/2}(\Omega)}\|g\|_{L^t(\Omega)} \le C_9\|f\|_{L^{r/2}(\Omega)}\|g\|_{L^\tau(\Omega)}$$
$$\le C_8C_9\|f\|_{L^{r/2}(\Omega)}\|g\|_{W^{2-s,q'}(\Omega)}.$$

Thus $f \in W^{s-2,q}(\Omega)$ and (6.6) holds with $C_6 \ge C_8C_9$. We have proved (6.6) for $s<2$. Using (6.2), (6.6) and (6.4)

$$\|A(\mathbf{u},\mathbf{u}) - A(\mathbf{v},\mathbf{v})\|_{W^{s-2,q}(\Omega)} \le C_1C_2m\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)}\|\mathbf{u}\|_{W^{s,q}(\Omega)}$$
$$+\|(|\mathbf{u}|-|\mathbf{v}|)\mathbf{v}\|_{W^{s-2,q}(\Omega)} \le C_1C_2m\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)}\|\mathbf{u}\|_{W^{s,q}(\Omega)}$$
$$+C_6\|(|\mathbf{u}|-|\mathbf{v}|)\mathbf{v}\|_{L^{r/2}(\Omega)} \le C_1C_2m\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)}\|\mathbf{u}\|_{W^{s,q}(\Omega)}$$
$$+C_6C_4^2m^2\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)}\|\mathbf{v}\|_{W^{s,q}(\Omega)}$$
$$\le (C_1C_2m + +C_6C_4^2m^2)\|\mathbf{u}-\mathbf{v}\|_{W^{s,q}(\Omega)}\left(\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|\mathbf{v}\|_{W^{s,q}(\Omega)}\right).$$

□

Lemma 6.4. *Let* $\Omega \subset \mathbb{R}^m$ *be a bounded domain with Lipschitz boundary,* $1 \le s < \infty$ *and* $1 < q < \infty$. *For* $\mathbf{u}, \mathbf{v} \in W^{s,q}(\Omega; \mathbb{R}^m)$ *define*

$$B(\mathbf{u}, \mathbf{v}) := (\mathbf{u} \cdot \nabla)\mathbf{v}.$$

Suppose that one of the following conditions is satisfied:

(1) $1 < s$ *and* $q > m/(s+1)$.

(2) $s = 1$, $q > 2m/(m+1)$. *If* $m/(m-1) < q < m$ *suppose that* $q \ge m/2$.

Then there exists a positive constant C *such that the following holds: If* $\mathbf{u}, \mathbf{v} \in W^{s,q}(\Omega; \mathbb{R}^m)$ *then* $B(\mathbf{u}, \mathbf{v}) \in W^{s-2,q}(\Omega; \mathbb{R}^m)$ *and*

$$\|B(\mathbf{u}, \mathbf{v})\|_{W^{s-2,q}(\Omega)} \le C\|\mathbf{u}\|_{W^{s,q}(\Omega)}\|\mathbf{v}\|_{W^{s,q}(\Omega)}, \tag{6.8}$$

$$\|B(\mathbf{u}, \mathbf{u}) - B(\mathbf{v}, \mathbf{v})\|_{W^{s-2,q}(\Omega)} \le C\|\mathbf{u} - \mathbf{v}\|_{W^{s,q}(\Omega)} \left(\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|\mathbf{v}\|_{W^{s,q}(\Omega)}\right).$$

Proof. Suppose first that $s > 1$ and $q > m/(s+1)$. Clearly, $\min(s, s-1) > s-2$. Moreover, $s + (s-1) - (s-2) = s+1 > m/q$. According to Proposition 7.1 there is a constant C such that (6.8) holds.

Suppose now that $s = 1$. Put $q' = q/(q-1)$. Suppose first that $q \ge m$. According to Lemma 2.2 there exist $r \in (q', \infty)$ and a constant C_1 such that

$$\|g\|_{L^r(\Omega)} \le C_1\|g\|_{W^{1,q'}(\Omega)} \qquad \forall g \in W^{1,q'}(\Omega). \tag{6.9}$$

Since $1/q + 1/q' = 1$ we have $1/q + 1/r < 1$. Thus there exists $t \in (1, \infty)$ such that $1/q + 1/r + 1/t = 1$. According to Lemma 2.2 there is a constant C_2 such that

$$\|f\|_{L^t(\Omega)} \le C_2\|f\|_{W^{1,q}(\Omega)} \qquad \forall f \in W^{1,q}(\Omega). \tag{6.10}$$

If $h \in L^q(\Omega)$, $g \in W^{1,q'}(\Omega)$ and $f \in W^{1,q}(\Omega)$ then Hölder's inequality forces

$$\left|\int_\Omega fhg \, dx\right| \le \|f\|_{L^t(\Omega)}\|h\|_{L^q(\Omega)}\|g\|_{L^r(\Omega)} \le C_1C_2\|f\|_{W^{1,q}(\Omega)}\|h\|_{L^q(\Omega)}\|g\|_{W^{1,q'}(\Omega)}.$$

So, $fh \in W^{-1,q}(\Omega)$ and

$$\|fh\|_{W^{-1,q}(\Omega)} \le C_1C_2\|f\|_{W^{1,q}(\Omega)}\|h\|_{L^q(\Omega)}. \tag{6.11}$$

If $\mathbf{u}, \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^m)$ then $B(\mathbf{u}, \mathbf{v}) \in W^{-1,q}(\Omega; \mathbb{R}^m)$ and (6.8) holds with $C \ge C_1C_2m^2$.

Suppose now that $s = 1$ and $q < m$. Put $t := mq/(m-q)$. According to Lemma 2.2 there exists a constant C_2 such that (6.10) hods. Since $q > 2m/(m+1)$ we have

$$\frac{1}{q} + \frac{1}{t} = \frac{1}{q} + \frac{m-q}{mq} < \frac{m+1}{2m} + \frac{m - 2m/(m+1)}{2m^2/(m+1)} = \frac{m+1}{2m} + \frac{m+1-2}{2m} = 1.$$

Therefore there is $r \in (1, \infty)$ such that $1/q + 1/t + 1/r = 1$. Hölder's inequality forces

$$\left|\int_\Omega fhg \, dx\right| \le \|f\|_{L^t(\Omega)}\|h\|_{L^q(\Omega)}\|g\|_{L^r(\Omega)}. \tag{6.12}$$

Suppose first that $q' = q/(q-1) \ge m$. According to Lemma 2.2 there exists a constant C_1 such that (6.9) holds. Suppose now that $q' < m$. Then $q = q'/(q'-1) > m/(m-1)$. So, $q \ge m/2$ by assumption. Thus

$$\frac{1}{r} - \frac{m-q'}{mq'} = 1 - \frac{1}{q} - \frac{1}{t} - \frac{m - q/(q-1)}{mq/(q-1)} = 1 - \frac{1}{q} - \frac{m-q}{mq} - \frac{mq-m-q}{mq}$$

$$= \frac{mq - m - m + q - mq + m + q}{mq} = \frac{2q - m}{mq} \geq 0.$$

Hence $r \leq mq'/(m - q')$. According to Lemma 2.2 there exists a constant C_1 such that (6.9) holds. According to (6.12), (6.9) and (6.10)

$$\left| \int_\Omega fgh \, dx \right| \leq \|f\|_{L^t(\Omega)} \|h\|_{L^q(\Omega)} \|g\|_{L^r(\Omega)} \leq C_1 C_2 \|f\|_{W^{1,q}(\Omega)} \|h\|_{L^q(\Omega)} \|g\|_{W^{1,q'}(\Omega)}.$$

So, if $f \in W^{1,q}(\Omega)$ and $h \in L^q(\Omega)$, then $fh \in W^{-1,q}(\Omega)$ and (6.11) holds. If $\mathbf{u}, \mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^m)$ then $B(\mathbf{u}, \mathbf{v}) \in W^{-1,q}(\Omega; \mathbb{R}^m)$ and (6.8) holds with $C \geq C_1 C_2 m^2$.

Clearly

$$\|B(\mathbf{u}, \mathbf{u}) - B(\mathbf{v}, \mathbf{v})\|_{W^{s-2,q}(\Omega)} = \|B(\mathbf{u} - \mathbf{v}, \mathbf{u}) + B(\mathbf{v}, \mathbf{u} - \mathbf{v})\|_{W^{s-2,q}(\Omega)} \leq$$
$$\|B(\mathbf{u} - \mathbf{v}, \mathbf{u})\|_{W^{s-2,q}(\Omega)} + \|B(\mathbf{v}, \mathbf{u} - \mathbf{v})\|_{W^{s-2,q}(\Omega)} \leq$$
$$C\|\mathbf{u} - \mathbf{v}\|_{W^{s,q}(\Omega)} \|\mathbf{u}\|_{W^{s,q}(\Omega)} + C\|\mathbf{v}\|_{W^{s,q}(\Omega)} \|\mathbf{u} - \mathbf{v}\|_{W^{s,q}(\Omega)}.$$

□

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 \leq s < \infty$ and $1 < q < \infty$. Suppose that one of the following conditions is satisfied:*

(1) $m \leq 4$, $s = 1$ and $q = 2$.
(2) $\Omega \subset \mathbb{R}^2$, $s = 1$ and $4/3 < q < 4$.
(3) $\Omega \subset \mathbb{R}^3$, $s = 1$ and $3/2 < q < 3$.
(4) $\partial\Omega$ is of class $\mathcal{C}^1$, $s = 1$ and $q > 2m/(m + 1)$. If $m/(m - 1) < q < m$ then $q \geq m/2$.
(5) $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ with $k \in N$, $1 < s \leq k + 1$ and $q > m/(s + 1)$.

Let $0 \leq \lambda, a, b, \beta < \infty$. If $s > 2$ or $q \leq m/3$ suppose that $a = 0$. If $m = 3$, $s = 1$ and $q < 6/5$ suppose that $a = 0$. Then there exist $\delta, \epsilon, C \in (0, \infty)$ such that the following holds: If $\mathbf{f} \in W^{s-2,q}(\Omega; \mathbb{R}^m)$, $\chi \in W^{s-1,q}(\Omega)$ and $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega; \mathbb{R}^m)$ satisfy

$$\|\mathbf{f}\|_{W^{s-2,q}(\Omega)} + \|\chi\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{s-1/q,q}(\partial\Omega)} < \delta \tag{6.13}$$

then there exists a solution $(\mathbf{u}, p) \in W^{s,q}(\Omega; \mathbb{R}^m) \times W^{s-1,q}(\Omega)$ of (1.3), (1.4) if and only if (5.2) holds. Moreover, there is a unique solution satisfying

$$\|\mathbf{u}\|_{W^{s,q}(\Omega)} < \epsilon \tag{6.14}$$

and (5.3). If $(\mathbf{u}, p)$ is a solution of (1.3), (1.4) satisfying (6.14) and (5.3) then

$$\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|p\|_{W^{s-1,q}(\Omega)} \leq C \left(\|\mathbf{f}\|_{W^{s-2,q}(\Omega)} + \|\chi\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{s-1/q,q}(\partial\Omega)} \right).$$

Proof. If $(\mathbf{u}, p) \in W^{s,q}(\Omega; \mathbb{R}^m) \times W^{s-1,q}(\Omega)$ is a solution of (1.3), (1.4), then (5.2) holds by Lemma 5.1.

Define

$$L(\mathbf{u}) := a|\mathbf{u}|\mathbf{u} + b(\mathbf{u} \cdot \nabla)\mathbf{u}.$$

According to Lemma 6.3 and Lemma 6.4 there is a constant C_1 such that

$$\|L\mathbf{u}\|_{W^{s-2,q}(\Omega)} \leq C_1 \|\mathbf{u}\|^2_{W^{s,q}(\Omega)},$$

$$\|L\mathbf{u} - L\mathbf{v}\|_{W^{s-2,q}(\Omega)} \leq C_1 \|\mathbf{u} - \mathbf{v}\|_{W^{s,q}(\Omega)} \left(\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|\mathbf{v}\|_{W^{s,q}(\Omega)} \right)$$

for all $\mathbf{u}, \mathbf{v} \in W^{s,q}(\Omega; \mathbb{R}^m)$. (If $m \leq 4$, $s = 1$ and $q = 2$ then $2m/(m + 1) < 2 = q$ and $m/2 \leq 2 = q$. If $m = 2$ and $4/3 < q < 4$ then $2m/(m + 1) = 4/3 < q$ and $m/(m - 1) = 2 = m$. If $m = 3$, $s = 1$ and $3/2 < q < 3$ then $2m/(m + 1) = 6/4 = 3/2 < q$ and $m/2 = 3/2 < q$. If $m = 3$, $1 < s < 2$ and $q > m/(s + 1)$

then $q > 3/(s+1) = 6/(2s+2) > 6/(3+2s)$. If $m \leq 3$ and $s = 1$ then $q > 1 \geq m/(2+s)$. If $m = 4$, $s = 1$ and $q = 2$ then $m/(2+s) = 4/3 < 2 = q$. If $s = 1$ and $m/(m-2+s) < q < m/s$, then $m/(m-1) < q < m$ and therefore $q \geq m/2 > m/(2+s)$.)

According to Theorem 5.4 there is a positive constant C_2 such that the following holds: If $\mathbf{f} \in W^{s-2,q}(\Omega;\mathbb{R}^m)$, $\chi \in W^{s-1,q}(\Omega)$ and $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega;\mathbb{R}^m)$ satisfy (5.2) then there is a unique solution $(\mathbf{u},p) \in W^{s,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega)$ of (1.1), (1.4) satisfying (5.3). Moreover,

$$\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|p\|_{W^{s-1,q}(\Omega)} \leq C_2 \left(\|\mathbf{f}\|_{W^{s-2,q}(\Omega)} + \|\chi\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{s-1/q,q}(\partial\Omega)}\right).$$

Put

$$\epsilon := \frac{1}{4(C_1+1)(C_2+1)}, \qquad \delta := \frac{\epsilon}{2(C_2+1)}.$$

If $(\mathbf{u},p) \in W^{s,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega)$ is a solution of (1.3), (1.4) satisfying (6.14) and (5.3), and $(\tilde{\mathbf{u}},\tilde{p}) \in W^{s,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega)$ is a solution of

$$-\Delta\tilde{\mathbf{u}} + \lambda\tilde{\mathbf{u}} + a|\tilde{\mathbf{u}}|\tilde{\mathbf{u}} + b(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} + \nabla\tilde{p} = \tilde{\mathbf{f}}, \ \nabla\cdot\tilde{\mathbf{u}} = \tilde{\chi} \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} + \beta\int_\Omega \tilde{\mathbf{u}}\ dx = \tilde{\mathbf{g}} \quad \text{on } \partial\Omega, \qquad \int_\Omega \tilde{p}\ dx = 0$$

and $\|\tilde{\mathbf{u}}\|_{W^{s,q}(\Omega)} < \epsilon$, then

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{s,q}(\Omega)} + \|p-\tilde{p}\|_{W^{s-1,q}(\Omega)} \leq C_2(\|\mathbf{f}-\tilde{\mathbf{f}}\|_{W^{s-2,q}(\Omega)} + \|\chi-\tilde{\chi}\|_{W^{s-1,q}(\Omega)}$$
$$+\|\mathbf{g}-\tilde{\mathbf{g}}\|_{W^{s-1/q,q}(\partial\Omega)} + \|L\mathbf{u}-L\tilde{\mathbf{u}}\|_{W^{s-2,q}(\Omega)}) \leq C_2(\|\mathbf{f}-\tilde{\mathbf{f}}\|_{W^{s-2,q}(\Omega)}$$
$$+\|\chi-\tilde{\chi}\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}-\tilde{\mathbf{g}}\|_{W^{s-1/q,q}(\partial\Omega)} + C_1 2\epsilon\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{s,q}(\Omega)})$$
$$\leq C_2(\|\mathbf{f}-\tilde{\mathbf{f}}\|_{W^{s-2,q}(\Omega)} + \|\chi-\tilde{\chi}\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}-\tilde{\mathbf{g}}\|_{W^{s-1/q,q}(\partial\Omega)}) + \frac{1}{2}\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{s,q}(\Omega)}.$$

Thus

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{s,q}(\Omega)} + \|p-\tilde{p}\|_{W^{s-1,q}(\Omega)}$$
$$\leq 2C_2(\|\mathbf{f}-\tilde{\mathbf{f}}\|_{W^{s-2,q}(\Omega)} + \|\chi-\tilde{\chi}\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}-\tilde{\mathbf{g}}\|_{W^{s-1/q,q}(\partial\Omega)}).$$

This gives the uniqueness of a solution of (1.3), (1.4) satisfying (6.14) and (5.3). For $\tilde{\mathbf{u}} \equiv 0$, $\tilde{p} \equiv 0$, $\tilde{\mathbf{f}} \equiv 0$, $\tilde{\chi} \equiv 0$, $\tilde{\mathbf{g}} \equiv 0$ we have

$$\|\mathbf{u}\|_{W^{s,q}(\Omega)} + \|p\|_{W^{s-1,q}(\Omega)} \leq 2C_2 \left(\|\mathbf{f}\|_{W^{s-2,q}(\Omega)} + \|\chi\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{s-1/q,q}(\partial\Omega)}\right).$$

Denote $E := \{\mathbf{u} \in W^{s,q}(\Omega;\mathbb{R}^m); \|\mathbf{u}\|_{W^{s,q}(\Omega)} \leq \epsilon\}$. Choose $\mathbf{f} \in W^{s-2,q}(\Omega;\mathbb{R}^m)$, $\chi \in W^{s-1,q}(\Omega)$ and $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega;\mathbb{R}^m)$ satisfying (6.13) and (5.2). For a fixed $\mathbf{v} \in E$ there exists a unique solution $(\mathbf{u}^\mathbf{v},p^\mathbf{v}) \in W^{s,q}(\Omega;\mathbb{R}^m) \times W^{s-1,q}(\Omega)$ of

$$-\Delta\mathbf{u}^\mathbf{v} + \lambda\mathbf{u}^\mathbf{v} + \nabla p^\mathbf{v} = \mathbf{f} - L(\mathbf{v}), \ \nabla\cdot\mathbf{u}^\mathbf{v} = \chi \quad \text{in } \Omega,$$

$$\mathbf{u}^\mathbf{v} + \beta\int_\Omega \mathbf{u}^\mathbf{v}\ dx = \mathbf{g} \quad \text{on } \partial\Omega, \qquad \int_\Omega p^\mathbf{v}\ dx = 0.$$

Clearly,

$$\|\mathbf{u}^\mathbf{v}\|_{W^{s,q}(\Omega)} \leq C_2\left(\|\mathbf{f}\|_{W^{s-2,q}(\Omega)} + \|L\mathbf{v}\|_{W^{s-2,q}(\Omega)} + \|\chi\|_{W^{s-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{s-1/q,q}(\partial\Omega)}\right)$$
$$< C_2\delta + C_2C_1\|\mathbf{v}\|^2_{W^{s,q}(\Omega)} = \frac{C_2\epsilon}{2(C_2+1)} + C_2C_1\epsilon\frac{1}{4(C_1+1)(C_2+1)} \leq \epsilon.$$

So $\mathbf{u}^\mathbf{v} \in E$. If $\mathbf{v},\mathbf{w} \in E$ then

$$\|\mathbf{u}^\mathbf{v} - \mathbf{u}^\mathbf{w}\|_{W^{s,q}(\Omega)} \leq C_2\|L\mathbf{v} - L\mathbf{w}\|_{W^{s-2,q}(\Omega)}$$
$$\leq C_2C_1\|\mathbf{w}-\mathbf{v}\|_{W^{s,q}(\Omega)}\left(\|\mathbf{w}\|_{W^{s,q}(\Omega)} + \|\mathbf{v}\|_{W^{s,q}(\Omega)}\right) \leq 2C_2C_1\epsilon\|\mathbf{w}-\mathbf{v}\|_{W^{s,q}(\Omega)}.$$

But $2C_2C_1\epsilon < 1$. Therefore Banach's fixed point theorem forces that there exists $\mathbf{v} \in E$ such that $\mathbf{u}^{\mathbf{v}} = \mathbf{v}$. (See [8, Satz 1/24].) For such $\mathbf{v}$ the pair $(\mathbf{u}^{\mathbf{v}}, p^{\mathbf{v}})$ is a solution of (1.3), (1.4). □

7. Appendix

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary. Let $0 < s(1), s(2) < \infty$, $\min(s(1), s(2)) \geq s > -\infty$ and $1 < p < \infty$. Suppose that $s(1) + s(2) - s > m/p$. Then there exists a positive constant C such that*

$$\|fg\|_{W^{s,p}(\Omega)} \leq C\|f\|_{W^{s(1),p}(\Omega)}\|g\|_{W^{s(2),p}(\Omega)}$$

for all $f \in W^{s(1),p}(\Omega)$, $g \in W^{s(2),p}(\Omega)$.

(See [24, Lemma 4.3].)

8. Declarations

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